

ON NONSYMMETRIC RANK ONE SINGULAR PERTURBATIONS OF SELFADJOINT OPERATORS

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ABSTRACT. We consider nonsymmetric rank one singular perturbations of a selfadjoint operator, i.e., an expression of the form $\tilde{A} = A + \alpha \langle \cdot, \omega_1 \rangle \omega_2$, $\omega_1 \neq \omega_2$, $\alpha \in \mathbb{C}$, in a general case $\omega_1, \omega_2 \in \mathcal{H}_{-2}$.

Using a constructive description of the perturbed operator \tilde{A} , we investigate some spectral and approximations properties of \tilde{A} . The wave operators corresponding to the couple A , \tilde{A} and a series of examples are also presented.

1. INTRODUCTION

The theory of rank one (symmetric) singular perturbations of a self adjoint operator has obtained much attention of physicists and mathematicians. Several papers and research monographs are devoted to this theory (see, f.e., [1, 3, 13] and references therein).

The aim of the current paper is to investigate a generalization of the singular (symmetric) perturbation theory to the case of nonsymmetric perturbations of the form $\tilde{A} = A + \alpha \langle \cdot, \omega_1 \rangle \omega_2$, where $A = A^*$ is a given selfadjoint operator perturbed by $\alpha \langle \cdot, \omega_1 \rangle \omega_2$ with vectors $\omega_1 \neq \omega_2$ that belong to the negative space \mathcal{H}_{-2} from the A -scale (the scale is generated by the operator A) and $\alpha \in \mathbb{C}$.

If $\omega_1 = \omega_2$ and $\alpha \in \mathbb{R}$, then we meet the classical case, i.e., the well known theory of a (symmetric) singular perturbation of selfadjoint operators [1, 3, 13].

In this article we continue our investigations started in [20], where we considered only the case $\omega_1, \omega_2 \in \mathcal{H}_{-1}$. A significant improvement of the previous studies is in the consideration of perturbations by arbitrary vectors, i.e., vectors from \mathcal{H}_{-2} , too.

Since the study of singularly perturbed operators is extended to perturbations by nonsymmetric potentials, we will also expect that the spectral properties (in particular the point spectrum of \tilde{A}) must be similar as in the classical case (including the unexpected for the perception the so-called associated pair of eigenvalues).

The next important topic of this article is to clarify approximation properties of these operators. In particular, we investigate an approximation of classical (symmetric) perturbations by nonsymmetric perturbations taking into account H_{-1} - and H_{-2} -perturbations.

When investigating wave operators for the couple A and a nonsymmetrically perturbed \tilde{A} , we solve a number of problems, including the following: whether \tilde{A} is a spectral type operator; whether \tilde{A} is additionally of a scalar type one; whether there exist wave operators for the couple A and \tilde{A} ; and whether we can write explicit expressions for these wave operators and, as a consequence, the scattering matrix.

In general, the idea and motivations for considering nonsymmetric perturbations is not new. Closely related investigations are in [15, 16], the ones carried out from the

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point of view of nonselfadjoint extensions in [19], and a non-local interactions approach has been taken in [17, 18].

2. THE MAIN DEFINITIONS AND PROPERTIES

Let \mathcal{H} be a separable Hilbert space with a scalar product (\cdot, \cdot) and the norm $\|\cdot\| = \sqrt{(\cdot, \cdot)}$. We consider a selfadjoint semi-bounded operator $A = A^*$ defined on a domain $\text{Dom} A = \mathfrak{D}(A)$ in \mathcal{H} . The sets $\sigma(\cdot)$, $\sigma_p(\cdot)$, $\sigma_{ac}(\cdot)$, $\sigma_c(\cdot)$, $\rho(\cdot)$ denote the spectrum, the point spectrum, the absolutely continuous spectrum, the continuous spectrum, and the regular points of a corresponding operator, respectively.

An operator A is associated with the A -scale of Hilbert spaces [3]. We consider only a part of the A -scale, namely,

$$(1) \quad \mathcal{H}_{-2} \supset \mathcal{H}_{-1} \supset \mathcal{H} \equiv \mathcal{H}_0 \supset \mathcal{H}_{+1} \supset \mathcal{H}_{+2}$$

where $\mathcal{H}_{+1} := \mathfrak{D}(|A|^{1/2})$ and $\mathcal{H}_{+2} := \mathfrak{D}(A)$ are endowed with the norms $\|\varphi\|_k = \|(|A| + I)^{k/2}\varphi\|$, $k = 1, 2$, $\varphi \in \mathcal{H}_k(A)$, respectively (I stands for the identity); and $\mathcal{H}_{-k} := \mathcal{H}_{-k}(A)$ is the negative (dual) space, i.e., the completion of \mathcal{H} with respect to the norm $\|f\|_{-k} = \|(|A| + I)^{-k/2}f\|$, $k = 1, 2$, $f \in \mathcal{H}$. Let $\langle \cdot, \cdot \rangle$ denote the usual dual scalar product for the spaces \mathcal{H}_k and \mathcal{H}_{-k} . The inner product in \mathcal{H}_k and \mathcal{H}_{-k} is denoted by $(\cdot, \cdot)_{\pm k}$, $k = 1, 2$.

The operator A has an extension by continuity to \mathcal{H} (\mathcal{H}_1) and it is understood as a bounded operator from \mathcal{H} (\mathcal{H}_1) into \mathcal{H}_{-2} (\mathcal{H}_{-1}). We denote such an extension by \mathbf{A} and $\mathbf{R}_z = (\mathbf{A} - z)^{-1}$, $z \in \rho(\mathbf{A})$, is a corresponding resolvent.

In some cases, we can continue the usual dual scalar product $\langle \cdot, \cdot \rangle$ to the case $\langle \omega, \phi \rangle$, where $\omega, \phi \in \mathcal{H}_{-2}$ (of course $\omega \neq \phi$) in the following way. For example, if we can decompose vectors $\omega = \omega_1 + \omega_2$ and $\phi = \phi_1 + \phi_2$ so that $\text{spsupp}(\omega_i) \subseteq \Pi_i$, $\text{spsupp}(\phi_i) \subseteq \Pi_i$, $i = 1, 2$, $\Pi \cap \Pi = \emptyset$ and $\omega_1 \in \mathcal{H}_{+2}$, $\omega_2 \in \mathcal{H}_{-2}$, $\phi_1 \in \mathcal{H}_{-2}$, $\phi_2 \in \mathcal{H}_{+2}$, then we can have $\langle \omega, \phi \rangle = \langle \omega_1, \phi_1 \rangle + \langle \omega_2, \phi_2 \rangle < \infty$, and $\langle \omega_1, \phi_2 \rangle = \langle \omega_2, \phi_1 \rangle = 0$, since $\text{spsupp}(\omega_1) \cap \text{spsupp}(\phi_2) = \emptyset$ and $\text{spsupp}(\omega_2) \cap \text{spsupp}(\phi_1) = \emptyset$. Here $\text{spsupp}(\cdot)$ denotes the spectral support of the corresponding vector in the sense of the operator \mathbf{A} . By definition (cf. [9]), for $\omega \in \mathcal{H}_{-2}$,

$$\begin{aligned} \text{spsupp}(\omega) &:= \{\lambda \in \mathbb{R} \mid \forall O_{\lambda, \varepsilon} \exists \psi \in C_0(\mathbb{R}) \cap L_2(\mathbb{R}, d\rho(\lambda)) : \\ &\quad \text{supp}(\psi) \subset O_{\lambda, \varepsilon} \text{ and } \int_{\mathbb{R}} \hat{\omega}(\lambda) \psi(\lambda) d\rho(\lambda) \neq 0\}, \end{aligned}$$

where $O_{\lambda, \varepsilon}$ is an ε -neighborhood of a point λ ; $C_0(\mathbb{R})$ is the set of continues functions with compact supports on \mathbb{R} ; $\hat{\omega}(\lambda)$ denotes the Fourier image of the vector ω . According to the central spectral theorem [8], the Fourier transform between \mathcal{H}_{-2} and $L_2(\mathbb{R}, d\rho(\lambda))$ takes the operator \mathbf{A} to the multiplication operator by the independent variable λ in the space $L_2(\mathbb{R}, d\rho(\lambda))$ with the Borel measure $d\rho(\lambda)$.

Let us consider an operator V in the A -scale, such that $\mathfrak{D}(V) \subseteq \mathcal{H}_{+k}$ and $\mathfrak{R}(V) \subseteq \mathcal{H}_{-k}$, $k = 1, 2$. In our case, $V = V^{\omega_1, \omega_2} = \langle \cdot, \omega_1 \rangle \omega_2$, $\omega_1, \omega_2 \in \mathcal{H}_{-k}$, $k = 1, 2$. Since the operator \mathbf{A} is bounded and acts from \mathcal{H}_0 into the whole \mathcal{H}_{-k} , $k = 1, 2$, the expression $\mathbf{A} + V$ is a bounded linear operator from \mathcal{H}_{+k} into \mathcal{H}_{-k} . Let us remark that due to [7] the adjoint operator $(\mathbf{A} + V)^+$ is correctly defined and acts also from \mathcal{H}_{+k} into \mathcal{H}_{-k} , $k = 1, 2$.

Now, the formal expression $A + \alpha \langle \cdot, \omega_1 \rangle \omega_2$ has a sense of an operator $\mathbf{A} + \langle \cdot, \omega_1 \rangle \omega_2$ defined on \mathcal{H}_{+k} , acting from \mathcal{H}_{+k} into \mathcal{H}_{-k} , $k = 1, 2$, and restricted to \mathcal{H} ,

$$(2) \quad A^{\omega_1, \omega_2} = (\mathbf{A} + \langle \cdot, \omega_1 \rangle \omega_2) \upharpoonright_{\mathcal{H}}.$$

In what follows, we write usually A instead of \mathbf{A} and hence, R_z will be used instead of \mathbf{R}_z .

Without loss of generality, supposing also that the operator A is strongly positive $A > 0$, we give a constructive definition of the operator A^{ω_1, ω_2} .

Definition 1. Let A be a positive self-adjoint operator on a separable Hilbert space \mathcal{H} , and consider A^{ω_1, ω_2} in (2) with $\omega_i \in \mathcal{H}_{-2} \setminus \mathcal{H}$, $\omega_1 \neq \omega_2$. We put $\eta_i = \mathbf{A}^{-1}\omega_i$, $i = 1, 2$.

I. The operator A^{ω_1, ω_2} is called singularly rank-one nonsymmetrically uniquely perturbed with respect to the operator A if $|(A^{1/2}\eta_2, A^{1/2}\eta_1)| < \infty$. Moreover,

$$(3) \quad \mathfrak{D}(A^{\omega_1, \omega_2}) = \left\{ \psi = \varphi + b\eta_2 \mid \varphi \in \mathfrak{D}(A), \ b = b(\varphi) = \frac{(A\varphi, \eta_1)}{1 + (A^{1/2}\eta_2, A^{1/2}\eta_1)} \right\}$$

in the case $(A^{1/2}\eta_2, A^{1/2}\eta_1) \neq -1$, and

$$(4) \quad \mathfrak{D}(A^{\omega_1, \omega_2}) = \mathfrak{D}_{\mathcal{H}_1} \dot{+} \{c\eta_2\}, \quad \mathfrak{D}_{\mathcal{H}_1} = \{\varphi \in \mathfrak{D}(A) \mid (A\varphi, \eta_1) = 0\}$$

in the case $(A^{1/2}\eta_2, A^{1/2}\eta_1) = -1$, (this fact we denote by $A^{\omega_1, \omega_2} \in \mathcal{P}(A)$).

II. The operator A^{ω_1, ω_2} is called singularly rank-one nonsymmetric parametrically perturbed with respect to the operator A if $(A^{1/2}\eta_2, A^{1/2}\eta_1)$ does not exist. Moreover,

$$(5) \quad \mathfrak{D}(A^{\omega_1, \omega_2}) = \left\{ \psi = \varphi + b\eta_2 \mid \varphi \in \mathfrak{D}(A), \right. \\ \left. b = b(\varphi) = \frac{(A\varphi, \eta_1)}{1 + \tau + (A^{1/2}(A^2 + 1)^{-1/2}\eta_2, A^{1/2}(A^2 + 1)^{-1/2}\eta_1)} \right\}$$

in the case $(A^{1/2}(A^2 + 1)^{-1/2}\eta_2, A^{1/2}(A^2 + 1)^{-1/2}\eta_1) \neq -\tau - 1$, where $\tau \in \mathbb{C}$ is a parameter, and

$$(6) \quad \mathfrak{D}(A^{\omega_1, \omega_2}) = \mathfrak{D}_{\mathcal{H}_1} \dot{+} \{c\eta_2\}, \quad \mathfrak{D}_{\mathcal{H}_1} = \{\varphi \in \mathfrak{D}(A) \mid (A\varphi, \eta_1) = 0\}$$

in the case $(A^{1/2}(A^2 + 1)^{-1/2}\eta_2, A^{1/2}(A^2 + 1)^{-1/2}\eta_1) = -\tau - 1$, (this fact we denote by $A^{\omega_1, \omega_2} \in \mathcal{P}_\tau(A)$)

The action of the perturbed operator is given by the rule

$$A^{\omega_1, \omega_2}\psi = A\varphi$$

in each case.

Remark 1. If $|(A^{1/2}\eta_2, A^{1/2}\eta_1)| < \infty$ in the second part of Definition 1, then taking $\tau = (A^{1/2}(A^2 + 1)^{-1/2}\eta_2, A^{1/2}(A^2 + 1)^{-1/2}\eta_1)$ we obtain the first part of Definition 1, that was considered also in [20].

Remark 2. If $(A^{1/2}\eta_2, A^{1/2}\eta_1)$ does not exist i.e., we have the second part of Definition 1, then still $|(A^{1/2}(A^2 + 1)^{-1/2}\eta_2, A^{1/2}(A^2 + 1)^{-1/2}\eta_1)| < \infty$, since $\omega_1, \omega_2 \in \mathcal{H}_{-2}$.

Remark 3. The defined in Definition 1 operator A^{ω_1, ω_2} can be described in the following way. A linear closed operator $A^{\omega_1, \omega_2} \neq A$ densely defined on \mathcal{H} is nonsymmetric singularly perturbed with respect to the operator A if both sets,

$$(7) \quad \mathfrak{D} = \{f \in \mathfrak{D}(A) \cap \mathfrak{D}(A^{\omega_1, \omega_2}) \mid Af = \tilde{A}f\},$$

$$(8) \quad \mathfrak{D}_* = \{f \in \mathfrak{D}(A) \cap \mathfrak{D}((A^{\omega_1, \omega_2})^*) \mid Af = \tilde{A}^*f\},$$

are dense in \mathcal{H} . In general $\tilde{A} \in \mathcal{P}_\tau(A)$.

It is clear that for each operator $A^{\omega_1, \omega_2} \in \mathcal{P}_\tau(A)$, there exist densely defined symmetric restrictions, i.e., operators $\dot{A} := A \upharpoonright \mathfrak{D}$ and $\dot{A}_* := A \upharpoonright \mathfrak{D}_*$ with nontrivial deficiency indices

$$\mathbf{n}^\pm(\dot{A}) = \dim \ker(\dot{A} \mp z)^* \neq 0, \quad \mathbf{n}^\pm(\dot{A}_*) = \dim \ker(\dot{A}_* \mp z)^* \neq 0, \quad z \in \rho(A).$$

(In this article we meet often the case where $\mathbf{n}^\pm(\dot{A}) = \mathbf{n}^\pm(\dot{A}_*) = 1$.)

If $\mathfrak{D} = \mathfrak{D}_*$ and $\tilde{A} = \tilde{A}^*$, then we are in the case of the usual abstract definition of singularly perturbed selfadjoint operators [3, 13] $\tilde{A} \in \mathcal{P}_s(A)$, that is, the definition given

above generalizes the known definition of a selfadjoint singular perturbation to the case of a nonselfadjoint one.

The operator defined above, A^{ω_1, ω_2} , has the following general properties.

Proposition 1. *For an arbitrary nonzero constant $a \in \mathbb{C}$ we have $A^{a\omega_1, \omega_2} = A^{\omega_1, \bar{a}\omega_2}$.*

Proof. From Definition 1 in both cases (3), (4) and (5), (6) it follows that $\mathfrak{D}(A^{a\omega_1, \omega_2}) = \mathfrak{D}(A^{\omega_1, \bar{a}\omega_2})$ and $A^{a\omega_1, \omega_2}\psi = A^{\omega_1, \bar{a}\omega_2}\psi = A\varphi$. \square

Proposition 2. *The adjoint operator $(A^{\omega_1, \omega_2})^*$ satisfies the identity*

$$(A^{\omega_1, \omega_2})^* = A^{\omega_2, \omega_1}.$$

Proof. For the proof we use the second part of Definition 1 for A^{ω_1, ω_2} and A^{ω_2, ω_1} and verify the identity

$$(9) \quad (A^{\omega_1, \omega_2} f_1, f_2) = (f_1, A^{\omega_2, \omega_1} f_2),$$

for $f_1 \in \mathfrak{D}(A^{\omega_1, \omega_2})$ and $f_2 \in \mathfrak{D}(A^{\omega_2, \omega_1})$ of the form $f_1 = \varphi_1 + b_1\eta_2$ and $f_2 = \varphi_2 + b_2\eta_1$, correspondingly, $\varphi_1, \varphi_2 \in \mathfrak{D}(A)$. The left-hand side of (9) is of the form

$$(10) \quad (A^{\omega_1, \omega_2} f_1, f_2) = (A^{\omega_1, \omega_2}(\varphi_1 + b_1\eta_2), (\varphi_2 + b_2\eta_1)) = (A\varphi_1, \varphi_2) + \bar{b}_2(A\varphi_1, \eta_1),$$

where

$$b_1 = b_1(\varphi_1) = \frac{(A\varphi_1, \eta_1)}{1 + \tau + (A^{1/2}(A^2 + 1)^{-1/2}\eta_2, A^{1/2}(A^2 + 1)^{-1/2}\eta_1)}.$$

The right-hand side of (9) is of the form

$$(11) \quad (f_1, A^{\omega_2, \omega_1} f_2) = ((\varphi_1 + b_1\eta_2), A^{\omega_2, \omega_1}(\varphi_2 + b_2\eta_1)) = (\varphi_1, A\varphi_2) + b_1(\eta_2, A\varphi_2),$$

where

$$b_2 = b_2(\varphi_2) = \frac{(A\varphi_2, \eta_2)}{1 + \tau + (A^{1/2}(A^2 + 1)^{-1/2}\eta_1, A^{1/2}(A^2 + 1)^{-1/2}\eta_2)}.$$

The statement of the proposition follows from the obvious equality of last terms from (10) and (11),

$$\begin{aligned} & \frac{\overline{(A\varphi_2, \eta_2)}}{1 + \tau + (A^{1/2}(A^2 + 1)^{-1/2}\eta_1, A^{1/2}(A^2 + 1)^{-1/2}\eta_2)} (A\varphi_1, \eta_1) \\ &= \frac{(A\varphi_1, \eta_1)}{1 + \tau + (A^{1/2}(A^2 + 1)^{-1/2}\eta_2, A^{1/2}(A^2 + 1)^{-1/2}\eta_1)} (\eta_2, A\varphi_2). \end{aligned}$$

The proof in case (5), i.e., $(A^{1/2}(A^2 + 1)^{-1/2}\eta_2, A^{1/2}(A^2 + 1)^{-1/2}\eta_1) \neq -\tau - 1$ is completed. The case (6), i.e., $(A^{1/2}(A^2 + 1)^{-1/2}\eta_2, A^{1/2}(A^2 + 1)^{-1/2}\eta_1) = -\tau - 1$ is also valid.

The cases (3), (4) are particular with respect to (5), (6). \square

3. THE DESCRIPTION OF A RANK-ONE NONSYMMETRIC SINGULAR PERTURBATION BY RESOLVENTS

In this section we consider the perturbed operator with a parameter $\alpha \in \mathbb{C}$ and denote it by $\tilde{A} = A + \alpha\langle \cdot, \omega_1 \rangle \omega_2$, where $\omega_i \in \mathcal{H}_{-2} \setminus \mathcal{H}$ and $\|\omega_i\|_{-1} = 1$, $i = 1, 2$. The set of such operators is also denoted by $\mathcal{P}_\tau(A)$. At the beginning, let us briefly remark that if $\tilde{A} \in \mathcal{P}_\tau(A)$ then for the adjoint operator, we have $\tilde{A}^* \in \mathcal{P}_\tau(A)$, which is also due to the investigations in [7] and Proposition 2.

Theorem 1. *For the resolvents $R_z = (A - z)^{-1}$ and $\tilde{R}_z = (\tilde{A} - z)^{-1}$ of operators $A = A^* > 1$ and $\tilde{A} \in \mathcal{P}_\tau(A)$ on a separable Hilbert space \mathcal{H} , a formula similar to M. Krein's formula holds for $z, \xi, \zeta \in \rho(A) \cap \rho(\tilde{A})$,*

$$(12) \quad \tilde{R}_z = R_z + b_z(\cdot, n_{\bar{z}})m_z,$$

with

$$(13) \quad n_z = (A - \xi)(A - z)^{-1}n_\xi, \quad m_z = (A - \zeta)(A - z)^{-1}m_\zeta,$$

where $n_z, m_z \in \mathcal{H} \setminus \mathcal{H}_{+2}$ and

$$(14) \quad b_z^{-1} - b_\xi^{-1} = (\xi - z)(m_\xi, n_{\bar{z}}).$$

The vectors n_z, m_z and the number b_z are connected with ω_1, ω_2 as follows:

$$(15) \quad n_z = R_z \omega_1, m_z = R_z \omega_2, \quad -b_z^{-1} = \alpha^{-1} + \tau + \langle (A^2 + 1)^{-1} \omega_2, (1 + \bar{z}A)R_{\bar{z}} \omega_1 \rangle,$$

where $\alpha \neq 0$.

The case $\alpha = 0$ can be also included in the considerations, since in case $\alpha = 0$, we put $b_z \equiv 0$ and obtain $\tilde{R}_z \equiv R_z$.

Proof. From the expression $\tilde{A} = A + \alpha \langle \cdot, \omega_1 \rangle \omega_2$ and for some $z \in \rho(A) \cap \rho(\tilde{A})$ we have $\tilde{A} - z = A - z + \alpha \langle \cdot, \omega_1 \rangle \omega_2$, and hence

$$(16) \quad (\tilde{A} - z)^{-1} = (A - z)^{-1} - \alpha \langle \cdot, (A - \bar{z})^{-1} \omega_1 \rangle (\tilde{A} - z)^{-1} \omega_2,$$

where $(A - z)^{-1}$ and $(\tilde{A} - z)^{-1}$ are considered as operators from \mathcal{H}_{-2} into \mathcal{H} . In the case of the first part of Definition 1 we can continue our proof as in [20]. In the case of the second (general) part of Definition 1 we need to consider the next expression following from (16):

$$(17) \quad (\tilde{A} - z)^{-1} = (A - z)^{-1} - \alpha \{ \langle A(A^2 + 1)^{-1/2}, A(A - \bar{z})^{-1} \omega_1 \rangle + \langle (A^2 + 1)^{-1}, (1 + \bar{z}A)(A - \bar{z})^{-1} \omega_1 \rangle \} (\tilde{A} - z)^{-1} \omega_2.$$

In particular, for $\omega_2 \in \mathcal{H}_{-2}$ we have

$$(\tilde{A} - z)^{-1} \omega_2 = (A - z)^{-1} \omega_2 - \alpha \{ \tau + \langle (A^2 + 1)^{-1} \omega_2, (1 + \bar{z}A)(A - \bar{z})^{-1} \omega_1 \rangle \} (\tilde{A} - z)^{-1} \omega_2.$$

where instead of $\langle \omega_2, A(A^2 + 1)^{-1} \omega_1 \rangle$ we write τ . Substituting

$$(\tilde{A} - z)^{-1} \omega_2 = \frac{\alpha}{1 + \alpha \{ \tau + \langle (A^2 + 1)^{-1} \omega_2, (1 + \bar{z}A)(A - \bar{z})^{-1} \omega_1 \rangle \}} (A - z)^{-1} \omega_2.$$

into (17) we get

$$(18) \quad (\tilde{A} - z)^{-1} = (A - z)^{-1} - \frac{1}{\alpha^{-1} + \tau + \langle (A^2 + 1)^{-1} \omega_2, (1 + \bar{z}A)(A - \bar{z})^{-1} \omega_1 \rangle} \times (\cdot, (A - \bar{z})^{-1} \omega_1) (A - z)^{-1} \omega_2.$$

where $\alpha \neq 0$. If we put

$$(19) \quad n_z = (A - z)^{-1} \omega_1, \quad m_z = (A - z)^{-1} \omega_2,$$

and

$$(20) \quad b_z^{-1} = -(\alpha^{-1} + \tau + \langle (A^2 + 1)^{-1} \omega_2, (1 + \bar{z}A)(A - \bar{z})^{-1} \omega_1 \rangle),$$

then we obtain (15).

Analogously, starting with $\tilde{A}^* = A + \bar{\alpha} \langle \cdot, \omega_2 \rangle \omega_1$, we also go to (19) and (20) in the equivalent form

$$\bar{b}_z^{-1} = -(\bar{\alpha}^{-1} + \bar{\tau} + \langle (A^2 + 1)^{-1} \omega_1, (1 + zA)(A - z)^{-1} \omega_2 \rangle),$$

Let us remark that if $\omega_1, \omega_2 \in \mathcal{H}_{-2} \setminus \mathcal{H}$ then $n_z, m_z \in \mathcal{H} \setminus \mathcal{H}_2$ (see, for example [14]), and, more precisely, if $\omega_1, \omega_2 \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$, then $n_z, m_z \in \mathcal{H} \setminus \mathcal{H}_{+1}$ and if $\omega_1, \omega_2 \in \mathcal{H}_{-1} \setminus \mathcal{H}$, then $n_z, m_z \in \mathcal{H}_{+1} \setminus \mathcal{H}_{+2}$.

By using notations (19) in the form $n_z = (A - z)^{-1} \omega_1$ and $n_\xi = (A - \xi)^{-1} \omega_1$ we obtain $\omega_1 = (A - z)n_z = (A - \xi)n_\xi$ and consequently the first expression in (13),

$$n_z = (A - \xi)(A - z)^{-1} n_\xi.$$

This expression makes sense in \mathcal{H} if we consider A on \mathcal{H} (but not \mathbf{A}). Analogously we obtain the second expression in (13). By a use of the Hilbert identity with (13), we obtain (14),

$$\begin{aligned} b_z^{-1} - b_\xi^{-1} &= -\langle (A^2 + 1)^{-1} \omega_2, (1 + \bar{z}A)(A - \bar{z})^{-1} \omega_1 \rangle + \langle (A^2 + 1)^{-1} \omega_2, (1 + \bar{\xi}A)(A - \bar{\xi})^{-1} \omega_1 \rangle \\ &= \langle (A^2 + 1)^{-1} \omega_2, ((1 + \bar{z}A)(A - \bar{\xi})^{-1} - (1 + \bar{\xi}A)(A - \bar{z})^{-1}) \omega_1 \rangle \\ &= (\xi - z) \langle (A - \xi)^{-1} \omega_2, (A - \bar{z})^{-1} \omega_1 \rangle \\ &= (\xi - z)(m_\xi, n_{\bar{z}}). \end{aligned}$$

This completes the proof. \square

Let us remark that it is possible that $b_z = \infty$, and it is so iff $z \in \sigma_p(\tilde{A})$, but (12) is also valid in such a case.

At the end of the article, we give an example that illustrates the following corollary from Theorem 1.

Corollary 1. *If the operators $A = A^*$ and $\tilde{A} \in \mathcal{P}(A)$ have inverses on a separable Hilbert space \mathcal{H} , i.e., $0 \in \rho(A) \cap \rho(\tilde{A})$, then (12), (13) and (14) have the form*

$$(21) \quad \tilde{A}^{-1} = A^{-1} + b_0(\cdot, n_0)m_0,$$

where

$$(22) \quad n_0 = A^{-1} \omega_1, \quad m_0 = A^{-1} \omega_2, \quad -b_0^{-1} = \alpha^{-1} + \langle \omega_2, A^{-1} \omega_1 \rangle.$$

Proof. The proof follows from Theorem 1 in two ways. The first one is to take into account that $|\langle \omega_2, A^{-1} \omega_1 \rangle| < \infty$ and repeat the proof of Theorem 1. The second one is to substitute $z = 0$ into (12), (13) and (14). \square

4. SPECTRAL PROPERTIES OF RANK ONE NONSYMMETRIC SINGULAR PERTURBATIONS

As a starting point, let us remark that the continuous spectrum $\sigma_c(A)$ of the operator A is unchanged by finite rank perturbations, i.e., $\sigma_c(A) = \sigma_c(\tilde{A})$, $\tilde{A} \in \mathcal{P}_\tau(A)$.

Theorem 2. *Let the perturbed operator $\tilde{A} \in \mathcal{P}_\tau(A)$ possess a new eigenvalue $\lambda \in \mathbb{C}$ comparing to A , i.e., there exist $\lambda \in \sigma_p(\tilde{A})$, $\lambda \notin \sigma_p(A)$. Then for the corresponding eigenvectors φ, ψ : $\tilde{A}\varphi = \lambda\varphi$ and $\tilde{A}^*\psi = \bar{\lambda}\psi$, the following relations hold true:*

$$(23) \quad (\lambda - z)b_z(\varphi, n_{\bar{z}}) = 1, \quad \varphi = (A - z)(A - \lambda)^{-1}m_z;$$

$$(24) \quad (\bar{\lambda} - \bar{z})\bar{b}_z(\psi, m_z) = 1, \quad \psi = (A - \bar{z})(A - \bar{\lambda})^{-1}n_{\bar{z}}.$$

Proof. The proof follows from Theorem 1. Let $\tilde{A}\varphi = \lambda\varphi$, i.e.,

$$\tilde{R}_z\varphi = R_z\varphi + b_z(\varphi, n_{\bar{z}})m_z = (\lambda - z)^{-1}\varphi.$$

Hence, $b_z(\varphi, n_{\bar{z}})m_z = [(\lambda - z)^{-1} - R_z]\varphi = (\lambda - z)^{-1}(A - \lambda)(A - z)^{-1}\varphi$,

$$(25) \quad (\lambda - z)b_z(\varphi, n_{\bar{z}})(A - z)(A - \lambda)^{-1}m_z = \varphi.$$

Multiplying the last expression by $n_{\bar{z}}$ we obtain that

$$(\lambda - z)b_z(\varphi, n_{\bar{z}})((A - z)(A - \lambda)^{-1}m_z, n_{\bar{z}}) = (\varphi, n_{\bar{z}}),$$

and, hence,

$$(26) \quad (\lambda - z)b_z(\varphi, n_{\bar{z}}) = 1.$$

We remark that (25) and (26) gives $\varphi = (A - z)(A - \lambda)^{-1}m_z$. This proves (23). Analogously, considering $\tilde{A}^*\psi = \bar{\lambda}\psi$ we can prove (24). \square

Corollary 2. *Let us put $z = 0$ in the case $\tilde{A} \in \mathcal{P}(A)$ under the conditions of Theorem 2. Then (23) and (24) have the form*

$$\lambda b_0(\varphi, n_0) = 1, \quad \varphi = A(A - \lambda)^{-1}m_0; \quad \bar{\lambda} \bar{b}_0(\psi, m_0) = 1, \quad \psi = A(A - \bar{\lambda})^{-1}n_0.$$

Proposition 3. *Let the perturbed operator $\tilde{A} \in \mathcal{P}(A)$ possess a new eigenvalue $\lambda \in \mathbb{C}$ in comparison with A and let the eigenvectors be φ and ψ , i.e., $\tilde{A}\varphi = \lambda\varphi$ and $\tilde{A}^*\psi = \bar{\lambda}\psi$ then the relations (25) and (26), in terms of ω_1, ω_2 , have a form*

$$(27) \quad \alpha \langle (A - \lambda)^{-1}\omega_2, \omega_1 \rangle = -1, \quad \varphi = (A - \lambda)^{-1}\omega_2;$$

$$(28) \quad \bar{\alpha} \langle (A - \bar{\lambda})^{-1}\omega_1, \omega_2 \rangle = -1, \quad \psi = (A - \bar{\lambda})^{-1}\omega_1.$$

Proof. Taking n_z and m_z as in (15), we obtain (27) and (28). The second way to prove is the following. Instead of (12) we take the representation $\tilde{A} = A + \alpha \langle \cdot, \omega_1 \rangle \omega_2$ and conduct similar conversions with the corresponding calculations. \square

5. THE INVERSE SPECTRAL PROBLEM FOR A RANK ONE NONSYMMETRIC SINGULAR PERTURBATION

If we regard the formulation of the Theorem 2 as a direct spectral problem, then we can consider the following Theorem as a corresponding inverse problem.

Theorem 3. *For a given positive selfadjoint operator $A = A^*$ on a separable Hilbert space \mathcal{H} and $\lambda \in \mathbb{C}$ and vectors $\varphi, \psi \in \mathcal{H} \setminus \mathcal{H}_{+1}$ ($\varphi, \psi \in \mathcal{H}_{+1} \setminus \mathcal{H}_{+2}$), there exist a unique $\tilde{A} \in \mathcal{P}_\tau(A)$ ($\tilde{A} \in \mathcal{P}(A)$) such that $\tilde{A}\varphi = \lambda\varphi$ and $\tilde{A}^*\psi = \bar{\lambda}\psi$. Moreover, the operator \tilde{A} is defined by (12) as follows:*

$$(29) \quad \tilde{R}_z = R_z + b_z(\cdot, n_{\bar{z}})m_z,$$

with

$$(30) \quad m_z = (A - \lambda)(A - z)^{-1}\varphi, \quad n_{\bar{z}} = (A - \bar{\lambda})(A - \bar{z})^{-1}\psi$$

and

$$(31) \quad b_z^{-1} = (\lambda - z)(\varphi, n_{\bar{z}}), \quad (\bar{b}_z^{-1} = (\bar{\lambda} - \bar{z})(\psi, m_z)).$$

In general, the proof of Theorem 3 does not differ from the proof of the similar Theorem in [20] and hence we give a sketch of the proof there.

Proof. Let us remark that if $\varphi, \psi \in \mathcal{H} \setminus \mathcal{H}_{+1}$, then there exists $\tilde{A} \in \mathcal{P}_\tau(A)$ (possibly $\tilde{A} \in \mathcal{P}(A)$) and if $\varphi, \psi \in \mathcal{H}_{+1} \setminus \mathcal{H}_{+2}$, then there exists exactly $\tilde{A} \in \mathcal{P}(A)$.

The proof needs a supplementary proposition that has a general character. One of them is known from [10].

Proposition 4. *Let there be given a (half-bounded) positive selfadjoint operator A with domain $\mathfrak{D}(A)$ in a separable Hilbert space \mathcal{H} . Then for an arbitrary vector $\eta \in \mathcal{H} \setminus \mathfrak{D}(A)$ and an arbitrary number $z \in \mathbb{C}$, $\text{Im}(z) \neq 0$, there exists a restriction \dot{A} of the operator A such that $\eta = n_z$ is its defect vector namely $(\dot{A} - z)^*n_z = 0$.*

This proposition is used for \dot{A} and \dot{A}_* .

Next we use the result which, in some sense, is inverse to the one pointed out in Theorem 1, — that is a formula similar to M. Krein's formula but from the perturbation point of view, i.e., the perturbation of the resolvent of the selfadjoint operator by a one-dimensional skew projection.

Proposition 5. *Let there be given a positive selfadjoint operator A on a separable Hilbert space \mathcal{H} . The operator-valued function*

$$(32) \quad \tilde{R}_z := (A - z)^{-1} + b_z(\cdot, n_{\bar{z}})m_z$$

- 1) is a resolvent of a closed operator, if n_z and b_z satisfy conditions (13) and (14) and $n_{\bar{z}}$, $(m_{\bar{z}})$ is not an eigenvector of $A - z$, $(A - \bar{z})$;
- 2) is the resolvent of a rank one singularly perturbed operator iff, additionally to the previous assumption, the inclusions $n_z, m_z \in \mathcal{H} \setminus \mathcal{H}_{+2}$ hold true.

Let us give also a sketch of the proof. The part 1) is verified due to Theorem 7.7.1 [11] as follows: \tilde{R}_z is the resolvent of a closed operator iff

- a) \tilde{R}_z satisfies the Hilbert identity $\tilde{R}_z - \tilde{R}_\xi = (z - \xi)\tilde{R}_z\tilde{R}_\xi$, $\text{Im} z, \xi \neq 0$;
- b) \tilde{R}_z has the trivial kernel, $\ker(\tilde{R}_z) = \{0\}$, $\text{Im} z \neq 0$.

The part a) is verified by substituting (32) into the Hilbert identity for the resolvent. The condition b) is verified by directly checking (32) for the vectors $f \perp n_{\bar{z}}$ and $n_{\bar{z}}$. The condition 2) follows from the fact that $n_z, m_z \in \mathcal{H} \setminus \mathfrak{D}(A)$.

We now return to the sketch of the proof of Theorem 3. Taking n_z, m_z in the form (30) and (31) we will check identity (14). Since the vectors n_z and m_z belong to $\mathcal{H} \setminus \mathcal{H}_{+2}$, by Proposition 5, i.e., by part 2), the operator \tilde{A} is singularly perturbed with respect of A . The identity $\tilde{A}\varphi = \lambda\varphi$ is checked by direct calculation. The uniqueness is proved by contradiction. \square

Proposition 6. *For a given selfadjoint operator A on a separable Hilbert space \mathcal{H} , and $\lambda \in \mathbb{C}$, and vectors $\varphi, \psi \in \mathcal{H}_{+1} \setminus \mathcal{H}_{+2}$, there exist a unique $\tilde{A} \in \mathcal{P}(A)$ such that $\tilde{A}\varphi = \lambda\varphi$, $\tilde{A}^*\psi = \bar{\lambda}\psi$. Moreover, the operator \tilde{A} is defined by the expression $\tilde{A} = A + \alpha\langle \cdot, \omega_1 \rangle \omega_2$, where $\omega_1 = (A - \bar{\lambda})\psi$, $\omega_2 = (A - \lambda)\varphi$, and $\alpha^{-1} = -\langle (A - \lambda)\omega_2, \omega_1 \rangle$, or $\alpha^{-1} = -\langle (A - \bar{\lambda})\omega_1, \omega_2 \rangle$.*

Proof. The corresponding proof is an implication of Theorems 1 and 3. \square

6. A DUAL PAIR OF EIGENVALUES

Since $\tilde{A} \in \mathcal{P}_\tau(A)$ is a non-self-adjoint operator, the definition of a dual pair of eigenvalues is different from [2].

Definition 2. *A couple of numbers $\lambda, \mu \in \mathbb{C}$ is called a dual pair of eigenvalues of a singularly perturbed operator $\tilde{A} \in \mathcal{P}_\tau(A)$ iff*

$$(33) \quad \tilde{A}\varphi_\lambda = \lambda\varphi_\lambda, \quad \tilde{A}\varphi_\mu = \mu\varphi_\mu,$$

$$(34) \quad \tilde{A}^*\psi_{\bar{\lambda}} = \bar{\lambda}\psi_{\bar{\lambda}}, \quad \tilde{A}^*\psi_{\bar{\mu}} = \bar{\mu}\psi_{\bar{\mu}},$$

$$(35) \quad (\bar{\lambda} - \bar{\mu})((A - \mu)^{-1}\varphi_\lambda, \psi_{\bar{\lambda}}) = (\varphi_\lambda, \psi_{\bar{\lambda}}).$$

The next theorem describes a method how to construct an operator with a dual pair.

Theorem 4. *Let $A = A^* \geq c$ be a semibounded selfadjoint operator defined on $\mathfrak{D}(A)$ in a separable Hilbert space \mathcal{H} . For an arbitrary $\mu \in \rho(A)$ and vectors $\varphi_\lambda, \psi_{\bar{\lambda}} \in \mathcal{H} \setminus \mathcal{H}_{+2}$ there exists a unique nonsymmetric singularly perturbed operator $\tilde{A} \in \mathcal{P}_\tau(A)$ such that (μ, λ) is a dual pair, where $\bar{\lambda} := \bar{\mu} + \frac{(\varphi_\lambda, \psi_{\bar{\lambda}})}{((A - \mu)^{-1}\varphi_\lambda, \psi_{\bar{\lambda}})}$ is an eigenvalue with the eigenvector $\varphi_\mu = (A - \lambda)(A - \mu)^{-1}\varphi_\lambda$. The adjoint operator $\tilde{A}^* \in \mathcal{P}_\tau(A)$ has also eigenvectors $\psi_{\bar{\lambda}}$ and $\psi_{\bar{\mu}} = (A - \bar{\lambda})(A - \bar{\mu})\psi_{\bar{\lambda}}$.*

Remark 4. If $\tilde{A} \in \mathcal{P}(A)$, then in the form $\tilde{A} = A + \alpha\langle \cdot, \omega_1 \rangle \omega_2$, we can calculate the coupling constant $\alpha = -\frac{1}{(\varphi_\lambda, \omega_1)}$ (or $\bar{\alpha} = -\frac{1}{(\psi_{\bar{\lambda}}, \omega_2)}$) and the corresponding vectors

$$(36) \quad \omega_2 = (A - \mu)\varphi_\lambda - \frac{(\psi_{\bar{\lambda}}, \varphi_\lambda)}{((A - \mu)\psi_{\bar{\lambda}}, \varphi_\lambda)}\varphi_\lambda, \quad \omega_1 = (A - \bar{\mu})\psi_{\bar{\lambda}} - \frac{(\varphi_\lambda, \psi_{\bar{\lambda}})}{((A - \mu)\varphi_\lambda, \psi_{\bar{\lambda}})}\psi_{\bar{\lambda}}.$$

Proof. The proof is given by a direct verification of only (33) and (34), since (35) is the condition of Theorem. \square

For a real dual pair $\lambda, \mu \in \mathbb{R}$ we have the following corollary that follows from Theorem 4.

Corollary 3. *Let $A = A^* \geq 0$ be a positive selfadjoint operator defined on $\mathfrak{D}(A)$ in a separable Hilbert space \mathcal{H} , so that $\sigma(A) = \sigma_c(A) = [0, \infty)$. For an arbitrary number $\mu < 0$ and vectors $\varphi_\lambda, \psi_{\bar{\lambda}} \in \mathcal{H}_{+1} \setminus \mathcal{H}_{+2}$ there exists a unique nonsymmetric singular perturbation of rank one, $\tilde{A} \in \mathcal{P}(A)$, such that it has the dual pair (μ, λ) , where $\lambda = \mu + \frac{(\varphi_\lambda, \psi_\lambda)}{((A-\mu)^{-1}\varphi_\lambda, \psi_\lambda)}$, as its eigenvalues with the eigenvectors φ_λ and $\varphi_\mu = (A - \lambda)(A - \mu)^{-1}\varphi_\lambda$. Moreover, the operator $\tilde{A}^* \in \mathcal{P}(A)$ has the same eigenvalues but with different eigenvectors, ψ_λ and $\psi_\mu = (A - \lambda)(A - \mu)\psi_\lambda$.*

7. APPROXIMATIONS PROPERTIES OF PERTURBED OPERATORS

Different approximations of singularly perturbed selfadjoint operators are presented in [4]. We have some generalization to the case of a nonsymmetric rank one perturbation.

Theorem 5. *Let A be a semibounded self-adjoint operator defined on $\mathfrak{D}(A)$ in a separable Hilbert space \mathcal{H} , and two vectors $\omega_1, \omega_2 \in \mathcal{H}_{-2}$ such that $\langle \omega_2, (A - z)^{-1}\omega_1 \rangle$ does not exist. Then there exist two sequences $\omega_{1,n}, \omega_{2,n} \in \mathcal{H}_{-2}$ converging to ω_i , $i = 1, 2$, correspondingly, such that the sequence of operators $\tilde{A}_n = A + \alpha \langle \cdot, \omega_{1,n} \rangle \omega_{2,n} \in \mathcal{P}(A)$ converge to the operator $\tilde{A} = A + \alpha \langle \cdot, \omega_1 \rangle \omega_2 \in \mathcal{P}_\tau(A)$ in the norm resolvent sense if*

$$\lim_{n \rightarrow \infty} \langle \omega_{2,n}, A(A^2 + 1)^{-1}\omega_{1,n} \rangle = \tau.$$

Proof. Without loss of generality let us assume that $A = A^* \geq 0$. Resolvents of the corresponding operators have the following forms:

$$(37) \quad (\tilde{A}_n - z)^{-1} = (A - z)^{-1} - \frac{1}{\alpha^{-1} + \langle \omega_{2,n}, (A - \bar{z})^{-1}\omega_{1,n} \rangle} \langle \cdot, (A - \bar{z})^{-1}\omega_{1,n} \rangle (A - z)^{-1}\omega_{2,n};$$

$$(38) \quad (\tilde{A} - z)^{-1} = (A - z)^{-1} - \frac{1}{\alpha^{-1} + \langle \omega_2, (1 + \bar{z}A)(A - \bar{z})^{-1}(A^2 + 1)^{-1}\omega_1 \rangle} \langle \cdot, (A - \bar{z})^{-1}\omega_1 \rangle (A - z)^{-1}\omega_2.$$

The difference of the resolvents has the form

$$\begin{aligned} & (\tilde{A}_n - z)^{-1} - (\tilde{A} - z)^{-1} = \\ & - \frac{1}{\alpha^{-1} + \langle \omega_{2,n}, (A - \bar{z})^{-1}\omega_{1,n} \rangle} \langle \cdot, (A - \bar{z})^{-1}\omega_{1,n} \rangle (A - z)^{-1}\omega_{2,n} \\ & + \frac{1}{\alpha^{-1} + \langle \omega_2, (1 + \bar{z}A)(A - \bar{z})^{-1}(A^2 + 1)^{-1}\omega_1 \rangle} \langle \cdot, (A - \bar{z})^{-1}\omega_1 \rangle (A - z)^{-1}\omega_2 \\ & = \left[- \frac{1}{\alpha^{-1} + \langle \omega_{2,n}, (A - \bar{z})^{-1}\omega_{1,n} \rangle} \right. \\ & \quad \left. + \frac{1}{\alpha^{-1} + \langle \omega_2, (1 + \bar{z}A)(A - \bar{z})^{-1}(A^2 + 1)^{-1}\omega_1 \rangle} \right] \\ & \times \langle \cdot, (A - \bar{z})^{-1}\omega_{1,n} \rangle (A - z)^{-1}\omega_{2,n} + \\ & \quad + \frac{1}{\alpha^{-1} + \langle \omega_2, (1 + \bar{z}A)(A - \bar{z})^{-1}(A^2 + 1)^{-1}\omega_1 \rangle} \\ & \times \langle \cdot, ((A - \bar{z})^{-1}\omega_{1,n} - (A - \bar{z})^{-1}\omega_1) \rangle (A - z)^{-1}\omega_2 \\ & \quad + \frac{1}{\alpha^{-1} + \langle \omega_2, (1 + \bar{z}A)(A - \bar{z})^{-1}(A^2 + 1)^{-1}\omega_1 \rangle} \end{aligned}$$

$$\times \langle \cdot, (A - \bar{z})^{-1} \omega_1 \rangle ((A - z)^{-1} \omega_{2,n} - (A - z)^{-1} \omega_2).$$

To prove that the resolvents of \tilde{A}_n converge to the resolvent \tilde{A} with respect to the operator norm it is enough to show that

$$(39) \quad \|(A - z)^{-1} \omega_{i,n} - (A - z)^{-1} \omega_i\| \longrightarrow 0, \quad i = 1, 2,$$

and

$$(40) \quad \langle \omega_{2,n}, (A - \bar{z})^{-1} \omega_{1,n} \rangle \longrightarrow \langle \omega_{2,n}, A(A^2 + 1)^{-1} \omega_{1,n} \rangle \\ + \tau + \langle \omega_{2,n}, (1 + \bar{z}A)(A - \bar{z})^{-1} (A^2 + 1)^{-1} \omega_{1,n} \rangle,$$

as $n \longrightarrow \infty$. Indeed, if $\omega_{i,n} \longrightarrow \omega_i$, in \mathcal{H}_{-2} , $i = 1, 2$, then (39) is valid.

Hence to prove the theorem it is sufficient to show that there exist sequences of vectors $\omega_{i,n}$, $i = 1, 2$, with the properties

$$\omega_{i,n} \longrightarrow \omega_i \text{ in } \mathcal{H}_{-2}, \text{ and } \lim_{n \rightarrow \infty} \langle \omega_{2,n}, A(A^2 + 1)^{-1} \omega_{1,n} \rangle = \tau.$$

Since the case $\omega_{i,n} \in \mathcal{H}_{-1}$ would be trivial, we consider the general case $\omega_{i,n} \in \mathcal{H}_{-2}$. Let us introduce the real sequence $a_n = \langle E_{[0,n]} \omega_2, A(A^2 + 1)^{-1} \omega_1 \rangle$, where E is the spectral measure of A . Since $\omega_i \notin \mathcal{H}_{-1}$, $i = 1, 2$, there exists an interval $[c_n, d_n]$ inside $[0, n]$, so that $b_n = \langle E_{[c_n, d_n]} \omega_2, A(A^2 + 1)^{-1} \omega_1 \rangle$ could be such that $|b_n| > |\tau - a_n|$. We choose the sequence $\omega_{i,n} = E_{[0,n]} \omega_i + \varepsilon_{i,n} E_{[c_n, d_n]} \omega_i$, $i = 1, 2$, where $\varepsilon_{i,n}$ are taken to be

$$\varepsilon_{1,n} = \frac{\sqrt{|\tau - a_n|}}{\sqrt{b_n}}, \quad \varepsilon_{2,n} = (\operatorname{sgn}(\tau - a_n) \operatorname{sgn} b_n) \varepsilon_{1,n}.$$

It is obvious that $|\varepsilon_{i,n}| \leq 1$ and $[\langle \omega_{2,n}, A(A^2 + 1)^{-1} \omega_{1,n} \rangle = a_n + \varepsilon_{1,n} \varepsilon_{2,n} b_n = \tau$. And we also have that $\|\omega_{i,n} - \omega_i\|_{-2} \leq \|E_{[0,\infty]} \omega_i\|_{-2} \longrightarrow 0$, as $n \longrightarrow \infty$. Hence \tilde{A}_n converges to \tilde{A} in the norm resolvent sense. \square

8. WAVE OPERATORS

Assuming that $i \notin \sigma(\tilde{A})$ we define the wave operators for A and $\tilde{A} \in \mathcal{P}_\tau(A)$ in a usual way [3],

$$(41) \quad W_\pm(A, \tilde{A}) = s - \lim_{t \rightarrow \pm\infty} U_t = s - \lim_{t \rightarrow \pm\infty} e^{i\tilde{A}t} e^{-iAt} P^{ac},$$

where

$$P^{ac} f = \int_{-\infty}^{\infty} \hat{f}^{ac}(\lambda) dE^{ac}(\lambda) (A + i)^{-1} \omega_1, \quad f^{ac} \in \mathcal{H}^{ac},$$

P^{ac} and \tilde{P}^{ac} denote the spectral projections onto the absolutely continuous parts $\sigma_{ac}(A)$ and $\sigma_{ac}(\tilde{A})$ of the spectrum of the operators A and \tilde{A} ; \mathcal{H}^{ac} and $\tilde{\mathcal{H}}^{ac}$ are the corresponding subspaces,

$$\mathcal{H}^{ac} = P^{ac} \mathcal{H}, \quad \tilde{\mathcal{H}}^{ac} = \tilde{P}^{ac} \mathcal{H};$$

$$f = \int_{-\infty}^{\infty} \hat{f}(\lambda) dE(\lambda) g = \int_{-\infty}^{\infty} \hat{f}(\lambda) d\tilde{E}(\lambda) \tilde{g}, \quad g = (A + i)^{-1} \omega_1, \quad \tilde{g} = (\tilde{A} + i)^{-1} \omega_2.$$

This definition of wave operators is correct due to Theorem 1.5 from [11], namely, \tilde{A} is the spectral type operator and what is more, with this connection \tilde{A} is the scalar type operator $\sigma_{ac}(A) = \sigma_{ac}(\tilde{A})$, and there exist wave operators for the couple A and \tilde{A} .

Theorem 6. *Let, on a separable Hilbert space \mathcal{H} , there be given a self-adjoint operator A and its nonsymmetric singular perturbation of the form*

$$\tilde{A} = A + \alpha \langle \cdot, \omega_1 \rangle \omega_2, \quad \omega_1, \omega_2 \in \mathcal{H}_{-2} \setminus \mathcal{H}, \quad \alpha \in \mathbb{C}.$$

Then there exist the defined in (41) wave operators W_{\pm} in the form:

$$(42) \quad W_{\pm} f = \int_{-\infty}^{\infty} (1 + \tau + \alpha F(\lambda \pm i0)) \hat{f}^{ac}(\lambda) d\tilde{E}(\lambda) (\tilde{A} + i)^{-1} \omega_2,$$

where

$$P^{ac} f = \int_{-\infty}^{\infty} \hat{f}^{ac}(\lambda) dE(\lambda) (A + i)^{-1} \omega_1, \quad F(z) = \langle (A - z)^{-1} \omega_2, (1 + \bar{z}A)(A^2 + I)^{-1} \omega_1 \rangle.$$

Proof. The proof can be carried out by a direct calculation as in [3] or by substituting the corresponding expressions in the formulas obtained in [20] and taking into account that $\tilde{A} \in \mathcal{P}_{\tau}(A)$ instead of $\tilde{A} \in \mathcal{P}(A)$. \square

The adjoint operator has the form

$$W_{+}^{*} g = \int_{-\infty}^{\infty} \frac{1}{1 + \tau + \alpha F(\lambda + i0)} \hat{g}^{ac}(\lambda) dE(A + i)^{-1} \omega_1,$$

where

$$\tilde{P}^{ac} g = \int_{-\infty}^{\infty} \hat{g}^{ac} d\tilde{E}(\lambda) (\tilde{A} + i)^{-1} \omega_2, \quad \tilde{g} = (\tilde{A} + i)^{-1} \omega_2 = \frac{1}{1 + \tau + F(-i)} g.$$

Then $S(\tilde{A}, A) = W_{+}^{*} W_{-}$ is a scattering operator and

$$(43) \quad S(\tilde{A}, A, \lambda, \tau) = \frac{1 + \tau + \alpha F(\lambda - i0)}{1 + \tau + \alpha F(\lambda + i0)}.$$

is a scattering matrix consisting of one (complex) number.

9. EXAMPLES

Example 1. Let us illustrate Theorems 1, 2, and 3. Let $\mathcal{H} = L_2([2, \infty), dx) = L_2$ and A be the operator of multiplication by the independent variable x^2 , namely

$$Af(x) = x^2 f(x), \quad \mathfrak{D}(A) = \{f(x) \in L_2 \mid x^2 f(x) \in L_2\}.$$

It is obvious that the operator $A \geq 2$ and A has purely absolutely continuous spectrum, i.e., $\sigma(A) = \sigma_{ac}(A) = [2, \infty)$.

Let us put $\mathcal{H}_1 = L_2([2, \infty), x^2 dx)$ and $\mathcal{H}_{-1} = L_2([2, \infty), x^{-2} dx)$ be the dual spaces. In such a case, $\mathfrak{D}(A) = \mathcal{H}_2 = L_2([2, \infty), x^4 dx)$, and $\mathcal{H}_{-2} = L_2([2, \infty), x^{-4} dx)$.

Let us take $\omega_1 = \frac{1}{x-1}$ and $\omega_2 = \frac{1}{x+1}$. It is obvious that $\omega_1, \omega_2 \in \mathcal{H}_{-1} \setminus \mathcal{H}$. So we can illustrate the operator $\tilde{A} \in \mathcal{P}(A)$ in the form (12).

If we suppose additionally that \tilde{A} possess a new point of spectrum $\lambda = 0$, i.e., $0 \in \sigma_p(\tilde{A})$, then due to Proposition 3 we calculate $\alpha = -\frac{2}{1-\ln 3}$ by the formula (27), since

$$\langle A^{-1} \omega_2, \omega_1 \rangle = \int_2^{\infty} \frac{dx}{x^2(x^2 - 1)} = \frac{1 - \ln 3}{2}.$$

Hence,

$$\tilde{A}f(x) = xf(x) - \frac{2}{1 - \ln 3} \frac{1}{x+1} \int_2^{\infty} \frac{f(x)}{x-1} dx.$$

Thus

$$\varphi = \frac{1}{x^2(x+1)}, \quad \psi = \frac{1}{x^2(x-1)}.$$

To illustrate Theorem 1, formulas (12) and Theorem 3, formulas (30), (31) (for the case where $\lambda = 0$) we must put

$$n_z = \frac{1}{(x^2 - z)(x - 1)}, \quad m_z = \frac{1}{(x^2 - z)(x + 1)}, \quad b_z^{-1} = \frac{2}{1 - \ln 3} - \int_2^{\infty} \frac{dx}{(x^2 - z)(x^2 - 1)}.$$

It is obvious that $\tilde{A} \in \mathcal{P}(A)$.

Example 2. This example illustrates Theorem 4. Let $\mathcal{H} = L_2(\mathbb{R}^1, dx)$ and A be a Laplace operator, namely $Af(x) = -f''(x)$, $\mathfrak{D}(A) = W_2^2(\mathbb{R}^1)$ is the Sobolev space. The operator $A \geq 0$ has purely absolutely continuous spectrum, namely $\sigma(A) = \sigma_c(A) = [0, \infty)$.

We put $\varphi_\lambda = e^{-|x-1|}$ and $\psi_\lambda = e^{-|x+1|}$, $\mu = -1$ (we consider the case $\lambda, \mu \in \mathbb{R}$ and $\lambda, \mu \in \rho(A)$). To calculate λ , we need

$$(\varphi_\lambda, \psi_\lambda) = \int_{\mathbb{R}} e^{-|x-1|} e^{-|x+1|} dx = 3e^{-2}.$$

By using [1], we calculate

$$(44) \quad (A+1)^{-1}\varphi_\lambda = \int_{\mathbb{R}} \frac{1}{2} e^{-|x-\tau|} e^{-|\tau-1|} d\tau = \begin{cases} \frac{x}{2} e^{-x+1}, & x > 1, \\ \frac{2-x}{2} e^{x-1}, & x < 1, \end{cases}$$

and $((A+1)^{-1}\varphi_\lambda, \psi_\lambda) = \frac{13}{4}e^{-2}$. Also, $\lambda = -\frac{1}{13} < 0$, $\alpha = -\frac{4}{13}e^2$. And from (36) we have that

$$\omega_1 = \delta_{-1}(x) - \frac{12}{13}e^{-|x+1|}, \quad \omega_2 = \delta_{+1}(x) - \frac{12}{13}e^{-|x-1|}.$$

The operator $\tilde{A} = A + \alpha\langle \cdot, \omega_1 \rangle \omega_2 \in \mathcal{P}(A)$ is such that $\tilde{A}\varphi_\lambda = \lambda\varphi_\lambda$, $\tilde{A}^*\psi_\lambda = \lambda\psi_\lambda$, $\tilde{A}\varphi_\mu = \mu\varphi_\mu$, $\tilde{A}^*\psi_\mu = \mu\psi_\mu$, and $\varphi_\mu = (A-\lambda)(A-\mu)^{-1}\varphi_\lambda = e^{-|x+1|}$, $\psi_\mu = e^{-|x-1|}$.

Example 3. We illustrate once more Theorem 1. Let $\mathcal{H} = L_2(\mathbb{R}^3)$ and A play the role of the Laplace operator, namely $Af(x) = -\Delta f(x)$, $\mathfrak{D}(A) = W_2^2(\mathbb{R}^3)$ is the Sobolev space. The operator is positive $A \geq 0$ and has purely absolutely continuous spectrum, i.e., $\sigma(A) = \sigma_c(A) = [0, \infty)$. Let us consider the expression (4) that describes the δ -interaction with retardation, i.e., the formal expression $\tilde{A} = -\Delta + \alpha\langle \cdot, \delta_0 \rangle \delta_1$, where δ_0 is a δ -function at the point $0 = (0, 0, 0) \in \mathbb{R}^3$ and δ_1 is a δ -function at the point $\mathbf{1} = (1, 0, 0) \in \mathbb{R}^3$.

Using [1] we can write the resolvent of such operators at the regular point $i \in \mathbb{C}$ (corresponding to one parameter family) i.e its integral kernel,

$$(-\tilde{\Delta} - i)^{-1}(x, p) = \frac{e^{-|x-p|}}{4\pi|x-p|} + \frac{1}{\alpha^{-1} + (4\pi e)^{-1}} \frac{e^{-(|x|+|\mathbf{1}-p|)}}{(4\pi)^2|x||\mathbf{1}-p|}$$

since $|\langle \delta_1, (A - \bar{z})^{-1}\delta_0 \rangle| = \frac{1}{4\pi e}$, $z = \sqrt{i}$, we take $(\Im \sqrt{i} > 0)$. It is obvious that $-\tilde{\Delta} \in \mathcal{P}(-\Delta)$.

Example 4. Let us again illustrate Theorem 4. Let $\mathcal{H} = L_2([1, \infty), dx)$ and A be the multiplication operator on x^2 , namely $Af(x) = x^2f(x)$, $f \in \mathfrak{D}(A)$, where $\mathfrak{D}(A) := \{f(x) \in L_2 \mid x^2f(x) \in L_2\}$. It is obvious that $A \geq 1$ and $\mathfrak{D}(A) = \sigma(A) = [1, \infty)$. We put $0 = \mu \notin \sigma(A)$ and $\varphi = \varphi_\lambda = x^{-1\frac{1}{3}}$, $\psi = \psi_\lambda = x^{-1\frac{2}{3}}$, $\varphi, \psi \in \mathcal{H}$ but $\varphi, \psi \notin \mathcal{H}_{+1}$. In particular $\mathcal{H}_{+2} = L_2([1, \infty), x^4 dx)$. Then

$$(\varphi, \psi) = \int_1^\infty \frac{dx}{x^3} = \frac{1}{2}, \quad ((A-\mu)^{-1}\varphi, \psi) = \int_1^\infty \frac{dx}{x^5} = \frac{1}{4}.$$

Hence, $\lambda = 2 \in \sigma(A)$, and

$$\varphi_\mu = (A-2)A^{-1}\varphi_\lambda = \frac{x^2-2}{x^{10/3}}, \quad \psi_\mu = (A-2)A^{-1}\psi_\lambda = \frac{x^2-2}{x^{11/3}}.$$

Also from (30) we have that

$$m_z = (A-\lambda)(A-z)^{-1}\varphi_\lambda = \frac{x^2-2}{x^2-z} \frac{1}{x^{4/3}}, \quad n_z = (A-\bar{\lambda})(A-\bar{z})^{-1}\psi_\lambda = \frac{x^2-2}{x^2-z} \frac{1}{x^{5/3}},$$

and from (31) we have $b_z = (\lambda - z)^{-1}(\varphi_\lambda, n_{\bar{z}})^{-1} = (2 - z)^{-1}(\varphi_\lambda, n_{\bar{z}})^{-1}$, where

$$(\varphi_\lambda, n_{\bar{z}}) = \int_1^\infty \frac{1}{x^{4/3}} \frac{x^2-2}{x^2-z} \frac{1}{x^{5/3}} dx = \left(\frac{1}{z^2} - \frac{1}{2z} \right) \ln \sqrt{1-z} + \frac{1}{z}.$$

Hence (29) has the form

$$(\tilde{A} - z)^{-1} = \frac{1}{x^2 - z} + b_z \left(\cdot, \frac{x^2 - 2}{x^2 - \bar{z}} \frac{1}{x^{5/3}} \right) \frac{x^2 - 2}{x^2 - z} \frac{1}{x^{4/3}}.$$

Moreover,

$$\omega_1 = \frac{x^2}{x^{5/3}} - \frac{1/2}{1/4} \frac{1}{x^{5/3}} = \frac{x^2 - 2}{x^{5/3}}, \quad \omega_2 = \frac{x^2}{x^{4/3}} - \frac{1/2}{1/4} \frac{1}{x^{4/3}} = \frac{x^2 - 2}{x^{4/3}}.$$

Since $\omega_1, \omega_2 \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$, $\tilde{A} \in \mathcal{P}_\tau(A)$.

Example 5. This example is a modification of the previous one and illustrates Corollary 3. Let $\mathcal{H} = L_2([1, \infty), dx)$ and, as above, A be an operator of multiplication by x^2 , namely $Af(x) = x^2 f(x)$, $f \in \mathfrak{D}(A)$, where $\mathfrak{D}(A) := \{f(x) \in L_2 \mid x^2 f(x) \in L_2\}$. For simplicity we also put $0 = \mu \notin \sigma(A)$, but $\varphi = \varphi_\lambda = x^{-2\frac{1}{3}}$, $\psi = \psi_\lambda = x^{-2\frac{2}{3}}$, $\varphi, \psi \in \mathcal{H}_{+1} = L_2([1, \infty), x^2 dx)$. In particular $\mathcal{H}_{+2} = L_2([1, \infty), x^4 dx)$ and $\varphi, \psi \notin \mathcal{H}_{+2}$. Then

$$(\varphi, \psi) = \int_1^\infty \frac{dx}{x^5} = \frac{1}{4}, \quad ((A - \mu)^{-1} \varphi, \psi) = \int_1^\infty \frac{dx}{x^7} = \frac{1}{6}.$$

Hence, $\lambda = 3/2 \in \sigma(A)$; and

$$\varphi_\mu = (A - 3/2)A^{-1}\varphi_\lambda = \frac{x^2 - 3/2}{x^{13/3}}, \quad \psi_\mu = (A - 3/2)A^{-1}\psi_\lambda = \frac{x^2 - 3/2}{x^{14/3}}.$$

And also from (30) we have that

$$m_z = (A - \lambda)(A - z)^{-1}\varphi_\lambda = \frac{x^2 - 3/2}{x^2 - z} \frac{1}{x^{7/3}}, \quad n_z = (A - \bar{\lambda})(A - \bar{z})^{-1}\psi_\lambda = \frac{x^2 - 3/2}{x^2 - z} \frac{1}{x^{8/3}},$$

and from (31) it follows that $b_z = (\lambda - z)^{-1}(\varphi_\lambda, n_{\bar{z}})^{-1} = (3/2 - z)^{-1}(\varphi_\lambda, n_{\bar{z}})^{-1}$, where

$$(\varphi_\lambda, n_{\bar{z}}) = \int_1^\infty \frac{1}{x^{7/3}} \frac{x^2 - 3/2}{x^2 - z} \frac{1}{x^{8/3}} dx = \left(\frac{3}{2z^3} - \frac{1}{z^2} \right) \ln \sqrt{1 - z} + \left(\frac{3}{4z^2} - \frac{1}{2z} \right) + \frac{3}{8z}.$$

Hence (29) has the form

$$(\tilde{A} - z)^{-1} = \frac{1}{x^2 - z} + b_z \left(\cdot, \frac{x^2 - 3/2}{x^2 - \bar{z}} \frac{1}{x^{8/3}} \right) \frac{x^2 - 3/2}{x^2 - z} \frac{1}{x^{7/3}}.$$

Moreover,

$$\omega_1 = \frac{x^2}{x^{8/3}} - \frac{1/4}{1/6} \frac{1}{x^{8/3}} = \frac{x^2 - 3/2}{x^{8/3}}, \quad \omega_2 = \frac{x^2}{x^{7/3}} - \frac{1/4}{1/6} \frac{1}{x^{7/3}} = \frac{x^2 - 3/2}{x^{7/3}}.$$

Since $\omega_1, \omega_2 \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$, $\tilde{A} \in \mathcal{P}_\tau(A)$, and we can exactly calculate the coupling constant.

$$\langle \varphi_\lambda, \omega_1 \rangle = \int_1^\infty \frac{1}{x^{7/3}} \frac{x^2 - 3/2}{x^{8/3}} dx = \langle \psi_\lambda, \omega_2 \rangle = \int_1^\infty \frac{1}{x^{8/3}} \frac{x^2 - 3/2}{x^{7/3}} dx = 1/8.$$

Due to (27), (28) we have that $\alpha = -\frac{1}{\langle \varphi_\lambda, \omega_1 \rangle} = -8$.

Example 6. This example once more illustrates Theorems 1, 2, and 3. Let $\mathcal{H} = L_2(\mathbb{R}^1)$ and A be also the Laplace operator, namely $Af(x) = -f''(x)$, $\mathfrak{D}(A) = W_2^2(\mathbb{R}^1)$ is the Sobolev space. The operator $A \geq 0$ is positive and has purely absolutely continuous spectrum, i.e., $\sigma(A) = \sigma_c(A) = [0, \infty)$. Let us consider the expression (4) that describes the δ -interaction with retardation on the real line.

Using [1] we can write the resolvent of such operators (corresponding to one parameter family) at a regular point k^2 , ($\Im k^2 > 0$), i.e its integral kernel,

$$(-\tilde{\Delta} - k^2)^{-1}(x, \xi) = (i/2k)e^{ik|x-\xi|} + \alpha(2k)^{-1}(i\alpha + 2k)^{-1}e^{ik[|x-x_2|+|x_1-\xi|]},$$

where

$$\Im k > 0, \quad \alpha \in \mathbb{C}, \quad x, \xi, x_1, x_2 \in \mathbb{R}^1, \quad x_1 < x_2.$$

It is not hard to understand that the essential spectrum is $\sigma_{ess}(-\tilde{\Delta}) = \sigma_{ac}(-\Delta) = [0, \infty)$, and, for the singularly continuous spectrum, we have $\sigma_{sc}(-\tilde{\Delta}) = \emptyset$. Moreover, if $\Re \alpha < 0$, then the operators $-\tilde{\Delta}$ (and $-\tilde{\Delta}^*$) possess precisely one negative, simple eigenvalue $\{-\alpha^2/4\}$ (and $\{-\bar{\alpha}^2/4\}$) with the corresponding normalized eigenfunction

$$\varphi = (-\alpha/2)^{1/2} e^{\alpha|x-y_1|/2}, \quad \psi = (-\bar{\alpha}/2)^{1/2} e^{\bar{\alpha}|x-y_2|/2}.$$

It is obvious that $-\tilde{\Delta} \in \mathcal{P}(-\Delta)$.

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